Modeling The Business Cycle
Mathematics Supplement

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In this white paper we will define and solve the equations that we will need to model the business cycle. We will be integrating over time so therefore we will define the variable t to be time in years.

We will define the following equations...

$$E_1 = \mathrm{Exp}\left\{\alpha\,t\right\} \;\Bigg|\; E_2 = \mathrm{Exp}\left\{\alpha\,t\right\}\sin(\beta\,(t+\phi)) \;\Bigg|\; E_3 = \mathrm{Exp}\left\{\alpha\,t\right\}\cos(\beta\,(t+\phi)) \;\Bigg|\; E_4 = \left(1+\Delta\,\sin(\beta\,\phi)\right)^{-1} \tag{1}$$

A. Using Equation (1) above we will make the following integral definition...

$$I(a,b)_1 = \int_a^b E_1\,\delta t = \int_a^b \mathrm{Exp}\left\{\alpha\,t\right\}\delta t \tag{2}$$

The solution to Equation (2) above is...

$$I(a,b)_1 = \mathrm{Exp}\left\{\alpha\,t\right\}\alpha^{-1}\Bigg[_a^b \tag{3}$$

If $b=\infty$ and $\alpha<0$ then using Equation (3) above the upper bound of Equation (2) is...

$$\text{upper bound} = 0 \;\text{ ...because... }\; \lim_{b\to\infty}\mathrm{Exp}\left\{\alpha\,b\right\} = 0 \tag{4}$$

B. Using Equation (1) above we will make the following integral definition...

$$I(a,b)_2 = \int_a^b E_2\,\delta t = \int_a^b \mathrm{Exp}\left\{\alpha\,t\right\}\sin(\beta\,(t+\phi))\,\delta t \tag{5}$$

The solution to Equation (5) above is...

$$I(a,b)_2 = \mathrm{Exp}\left\{\alpha\,t\right\}\left(\alpha\,\sin(\beta\,(t+\phi)) - \beta\,\cos(\beta\,(t+\phi))\right)\left(\alpha^2+\beta^2\right)^{-1}\Bigg[_a^b \tag{6}$$

If $b=\infty$ and $\alpha<0$ then using Equation (6) above the upper bound of Equation (5) is...

$$\text{upper bound} = 0 \;\text{ ...because... }\; \lim_{b\to\infty}\mathrm{Exp}\left\{\alpha\,b\right\} = 0 \tag{7}$$

C. Using Equation (1) above we will make the following integral definition...

$$I(a,b)_3 = \int_a^b E_3\,\delta t = \int_a^b \mathrm{Exp}\left\{\alpha\,t\right\}\cos(\beta\,(t+\phi))\,\delta t \tag{8}$$

The solution to Equation (8) above is...

$$I(a,b)_3 = \mathrm{Exp}\left\{\alpha\, t\right\}\left(\beta\,\sin(\beta\,(t+\phi)) + \alpha\,\cos(\beta\,(t+\phi))\right)\left(\alpha^2+\beta^2\right)^{-1}\Bigg[_a^b \tag{9}$$

If $b=\infty$ and $\alpha<0$ then using Equation (9) above the upper bound of Equation (8) is...

$$\text{upper bound} = 0 \;\;\text{...because...}\;\; \lim_{b\to\infty}\mathrm{Exp}\left\{\alpha\, b\right\} = 0 \tag{10}$$

D. Using Equation (1) above we will make the following derivative definition...

$$\frac{\delta}{\delta t}\,E_1 = \frac{\delta}{\delta t}\,\mathrm{Exp}\left\{\alpha\, t\right\} = \alpha\,\mathrm{Exp}\left\{\alpha\, t\right\} \tag{11}$$

Using Equation (11) above we can make the following statement...

$$\mathrm{Exp}\left\{\alpha\, b\right\} - \mathrm{Exp}\left\{\alpha\, a\right\} = \int_a^b \alpha\,\mathrm{Exp}\left\{\alpha\, t\right\}\delta t \tag{12}$$

E. Using Equation (1) above we will make the following derivative definition...

$$\frac{\delta}{\delta t}\,E_2 = \frac{\delta}{\delta t}\,\mathrm{Exp}\left\{\alpha\, t\right\}\,\sin(\beta\,(t+\phi)) = \mathrm{Exp}\left\{\alpha\, t\right\}\left(\alpha\,\sin(\beta\,(t+\phi)) + \beta\,\cos(\beta\,(t+\phi))\right) \tag{13}$$

Using Equation (13) above we can make the following statement...

$$\mathrm{Exp}\left\{\alpha\, b\right\}\,\sin(\beta\,(b+\phi)) - \mathrm{Exp}\left\{\alpha\, a\right\}\,\sin(\beta\,(a+\phi)) = \int_a^b \mathrm{Exp}\left\{\alpha\, t\right\}\left(\alpha\,\sin(\beta\,(t+\phi)) + \beta\,\cos(\beta\,(t+\phi))\right)\delta t \tag{14}$$

F. Using Equation (1) above we will make the following derivative definition...

$$\frac{\delta}{\delta t}\,E_3 = \frac{\delta}{\delta t}\,\mathrm{Exp}\left\{\alpha\, t\right\}\,\cos(\beta\,(t+\phi)) = \mathrm{Exp}\left\{\alpha\, t\right\}\left(\alpha\,\cos(\beta\,(t+\phi)) - \beta\,\sin(\beta\,(t+\phi))\right) \tag{15}$$

Using Equation (15) above we can make the following statement...

$$\mathrm{Exp}\left\{\alpha\, b\right\}\,\cos(\beta\,(b+\phi)) - \mathrm{Exp}\left\{\alpha\, a\right\}\,\cos(\beta\,(a+\phi)) = \int_a^b \mathrm{Exp}\left\{\alpha\, t\right\}\left(\alpha\,\cos(\beta\,(t+\phi)) - \beta\,\sin(\beta\,(t+\phi))\right)\delta t \tag{16}$$

G. Using Equation (1) above we redefine equation E_4 as...

$$E_4 = \left(1+\Delta\,\sin(\beta\,\phi)\right)^{-1} = A \;\;\text{...where...}\;\; A = B^{-1} \;\;\text{...and...}\;\; B = 1+\Delta\,\sin(\beta\,\phi) \tag{17}$$

Using Equation (17) above we will define the following derivative equations...

$$\frac{\delta A}{\delta B} = -B^{-2} = -\left(1+\Delta\,\sin(\beta\,\phi)\right)^{-2} \;\;\text{...where...}\;\; \frac{\delta B}{\delta\Delta} = \sin(\beta\,\phi) \tag{18}$$

Using Equations (17) and (18) above the first derivative of equation E_4 above with respect to the variable Δ is...

$$\frac{\delta}{\delta\Delta}\,E_4 = \frac{\delta A}{\delta B}\,\frac{\delta B}{\delta\Delta} = -\sin(\beta\,\phi)\left(1+\Delta\,\sin(\beta\,\phi)\right)^{-2} \tag{19}$$

Using Equations (18) and (19) above as our guide the second derivative of equation E_4 above with respect to the variable Δ is...

$$\frac{\delta}{\delta\Delta^2}\,E_4 = \frac{\delta A}{\delta B}\,\frac{\delta B}{\delta\Delta} = 2\,\sin(\beta\,\phi)^2\left(1+\Delta\,\sin(\beta\,\phi)\right)^{-3} \tag{20}$$